

Quarkonium Wave Functions at the Origin

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Abstract

We tabulate values of the radial Schrödinger wave function or its first nonvanishing derivative at zero quark-antiquark separation, for $c\bar{c}$, $c\bar{b}$, and $b\bar{b}$ levels that lie below, or just above, flavor threshold. These quantities are essential inputs for evaluating production cross sections for quarkonium states.

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Fragmentation of partons into quarkonium states recently has been recognized as an important component of quarkonium production in high-energy collisions [1]. Thorough investigation of this new source, and others, is made timely by the CDF Collaboration's observation that conventional sources substantially underestimate the yield of prompt J/ψ and ψ' in 1.8-TeV $\bar{p}p$ collisions [2]. In the case of the ψ' , which is not fed by the cascade decay of the narrow χ_c states, the observed cross section exceeds the calculated one by more than an order of magnitude.

Calculation of the production rate by fragmentation can be separated into a parton-level piece that can be evaluated using perturbative techniques and a hadronic piece expressed in terms of the quarkonium wave function. We have earlier tabulated the values at the origin of the radial wave function, or its first nonvanishing derivative, for narrow levels of the $b\bar{c}$ (B_c) system [3]. These have been used to estimate the B_c production rate [4]. Here we present the corresponding information for the $c\bar{c}$ (J/ψ) and $b\bar{b}$ (Υ) families. Although many of these numbers have appeared in the literature when quarkonium spectroscopy was in flower, they usually were given implicitly in calculations of leptonic widths or similar observables. Our purpose here is to record the relevant properties of the wave functions of all the narrow levels in a form convenient for evaluating cross sections for quarkonium production.

We consider four functional forms for the potential that give reasonable accounts of the $c\bar{c}$ and $b\bar{b}$ spectra: the QCD-motivated potential given by Buchmüller and Tye [5], with

$$m_c = 1.48 \text{ GeV}/c^2 \quad m_b = 4.88 \text{ GeV}/c^2 \quad ; \quad (1)$$

a power-law potential [6],

$$V(r) = -8.064 \text{ GeV} + (6.898 \text{ GeV})(r \cdot 1 \text{ GeV})^{0.1} \quad , \quad (2)$$

with

$$m_c = 1.8 \text{ GeV}/c^2 \quad m_b = 5.174 \text{ GeV}/c^2 \quad ; \quad (3)$$

a logarithmic potential [7],

$$V(r) = -0.6635 \text{ GeV} + (0.733 \text{ GeV}) \log(r \cdot 1 \text{ GeV}) \text{ ,} \quad (4)$$

with

$$m_c = 1.5 \text{ GeV}/c^2 \quad m_b = 4.906 \text{ GeV}/c^2 \text{ ;} \quad (5)$$

and a Coulomb-plus-linear potential (the ‘‘Cornell potential’’) [8],

$$V(r) = -\frac{\kappa}{r} + \frac{r}{a^2} \text{ ,} \quad (6)$$

with

$$m_c = 1.84 \text{ GeV}/c^2 \quad m_b = 5.18 \text{ GeV}/c^2 \quad (7)$$

$$\kappa = 0.52 \quad a = 2.34 \text{ GeV}^{-1} \text{ .} \quad (8)$$

For quarks bound in a central potential, it is convenient to separate the Schrödinger wave function into radial and angular pieces as $\Psi_{n\ell m}(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\theta, \phi)$, where n is the principal quantum number, ℓ and m are the orbital angular momentum and its projection, $R_{n\ell}(r)$ is the radial wave function, and $Y_{\ell m}(\theta, \phi)$ is a spherical harmonic [9]. The Schrödinger wave function is normalized, $\int d^3\vec{r} |\Psi_{n\ell m}(\vec{r})|^2 = 1$, so that $\int_0^\infty r^2 dr |R_{n\ell}(r)|^2 = 1$. The value of the radial wave function, or its first nonvanishing derivative at the origin,

$$R_{n\ell}^{(\ell)}(0) \equiv \left. \frac{d^\ell R_{n\ell}(r)}{dr^\ell} \right|_{r=0} \text{ ,} \quad (9)$$

is required to evaluate production rates through parton fragmentation. The quantity $|R_{n\ell}^{(\ell)}(0)|^2$ is presented for four potentials in Table I for the narrow charmonium levels and in Table II for the narrow Upsilon levels. For ease of reference, we reproduce in Table III our predictions [3] for the B_c wave functions, with some computational improvements in the entries for the Cornell potential. The strong Coulomb singularity of the Cornell potential is reflected in spatially smaller states.

In view of the efforts [10] to resolve the ψ' anomaly with cascades from above-threshold states, we quote values for the $c\bar{c}$ 3D, 3P, and 3S levels that lie near 3.8, 3.9, and 4.0 GeV/ c^2 , respectively, and for the $b\bar{b}$ 4F, 4D, 4P, and 4S levels that lie near 10.35, 10.45, 10.52, and

10.6 GeV/ c^2 , respectively. It is likely that these will be significantly modified by coupled-channel effects [8,11].

If all the potentials describe the $c\bar{c}$ and $b\bar{b}$ observables, what accounts for the wide variation in the values of $|R_{n0}(0)|^2$ and the corresponding quantities for orbital excitations? The leptonic decay rate of a neutral ($Q\bar{Q}$) vector meson V^0 is related to the Schrödinger wave function through [12,13]

$$\Gamma(V^0 \rightarrow e^+e^-) = \frac{4N_c\alpha^2 e_Q^2}{3} \frac{|R_{n0}(0)|^2}{M_V^2} \left(1 - \frac{16\alpha_s}{3\pi}\right) , \quad (10)$$

where $N_c = 3$ is the number of quark colors, e_Q is the heavy-quark charge, and M_V is the mass of the vector meson. The QCD correction reduces the magnitudes significantly; the amount of this reduction is somewhat uncertain, because the first term in the perturbation expansion is large [14]. Each of the potentials we use corresponds to a particular interpretation of the radiative correction to the leptonic width. Similar effects may enter the connection between wave functions and fragmentation probabilities.

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TABLES

TABLE I. Radial wave functions at the origin and related quantities for $c\bar{c}$ mesons.

Level	$ R_{n\ell}^{(\ell)}(0) ^2$			
	QCD (B–T), Ref. [5]	Power-law, Ref. [6]	Logarithmic, Ref. [7]	Cornell, Ref. [8]
1S	0.810 GeV ³	0.999 GeV ³	0.815 GeV ³	1.454 GeV ³
2P	0.075 GeV ⁵	0.125 GeV ⁵	0.078 GeV ⁵	0.131 GeV ⁵
2S	0.529 GeV ³	0.559 GeV ³	0.418 GeV ³	0.927 GeV ³
3D	0.015 GeV ⁷	0.026 GeV ⁷	0.012 GeV ⁷	0.031 GeV ⁷
3P	0.102 GeV ⁵	0.131 GeV ⁵	0.076 GeV ⁵	0.186 GeV ⁵
3S	0.455 GeV ³	0.410 GeV ³	0.286 GeV ³	0.791 GeV ³

TABLE II. Radial wave functions at the origin and related quantities for $b\bar{b}$ mesons.

Level	$ R_{n\ell}^{(\ell)}(0) ^2$			
	QCD (B-T), Ref. [5]	Power-law, Ref. [6]	Logarithmic, Ref. [7]	Cornell, Ref. [8]
1S	6.477 GeV ³	4.591 GeV ³	4.916 GeV ³	14.05 GeV ³
2P	1.417 GeV ⁵	1.572 GeV ⁵	1.535 GeV ⁵	2.067 GeV ⁵
2S	3.234 GeV ³	2.571 GeV ³	2.532 GeV ³	5.668 GeV ³
3D	0.637 GeV ⁷	0.892 GeV ⁷	0.765 GeV ⁷	0.860 GeV ⁷
3P	1.653 GeV ⁵	1.660 GeV ⁵	1.513 GeV ⁵	2.440 GeV ⁵
3S	2.474 GeV ³	1.858 GeV ³	1.736 GeV ³	4.271 GeV ³
4F	0.414 GeV ⁹	0.627 GeV ⁹	0.456 GeV ⁹	0.563 GeV ⁹
4D	1.191 GeV ⁷	1.396 GeV ⁷	1.119 GeV ⁷	1.636 GeV ⁷
4P	1.794 GeV ⁵	1.593 GeV ⁵	1.377 GeV ⁵	2.700 GeV ⁵
4S	2.146 GeV ³	1.471 GeV ³	1.324 GeV ³	3.663 GeV ³
5F	1.075 GeV ⁹	1.302 GeV ⁹	0.894 GeV ⁹	1.520 GeV ⁹
5D	1.722 GeV ⁷	1.689 GeV ⁷	1.289 GeV ⁷	2.417 GeV ⁷
5P	1.935 GeV ⁵	1.504 GeV ⁵	1.252 GeV ⁵	2.917 GeV ⁵
5S	1.956 GeV ³	1.231 GeV ³	1.077 GeV ³	3.319 GeV ³

TABLE III. Radial wave functions at the origin and related quantities for $c\bar{b}$ mesons.

Level	$ R_{n\ell}^{(\ell)}(0) ^2$			
	QCD (B–T), Ref. [5]	Power-law, Ref. [6]	Logarithmic, Ref. [7]	Cornell, Ref. [8]
1S	1.642 GeV ³	1.710 GeV ³	1.508 GeV ³	3.184 GeV ³
2P	0.201 GeV ⁵	0.327 GeV ⁵	0.239 GeV ⁵	0.342 GeV ⁵
2S	0.983 GeV ³	0.950 GeV ³	0.770 GeV ³	1.764 GeV ³
3D	0.055 GeV ⁷	0.101 GeV ⁷	0.055 GeV ⁷	0.102 GeV ⁷
3P	0.264 GeV ⁵	0.352 GeV ⁵	0.239 GeV ⁵	0.461 GeV ⁵
3S	0.817 GeV ³	0.680 GeV ³	0.563 GeV ³	1.444 GeV ³